

MEMORANDUM

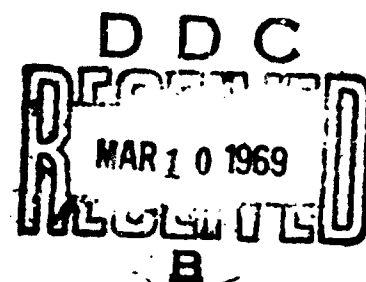
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# INVERSE FUNCTIONS OF THE PRODUCTS OF TWO BESSEL FUNCTIONS

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PREPARED FOR:

UNITED STATES AIR FORCE PROJECT RAND

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PREFACE AND SUMMARY

This Memorandum is a part of RAND's continuing study on electromagnetic propagation as affected by atmospheric scattering. Atmospheric properties have been inferred from scattering coefficients, whose calculations involve finding certain inverse operators. Special cases of the inverse function of the product of two spherical Bessel functions have been found recently by other writers as  ${}_1F_2$  hypergeometric functions. This Memorandum gives the general expression as the derivative of a product of spherical Bessel functions. These results may have application in various scattering configurations, and also to the classical physics problem of determining the potential from the phase shift.

# INVERSE FUNCTIONS OF THE PRODUCTS OF TWO BESSEL FUNCTIONS

A recent article<sup>(1)</sup> considers the problem of finding the inverse function of the product of two spherical Bessel functions. The results are given in terms of  ${}_1F_2$  hypergeometric functions. The inverse function may be defined by the equation

$$\int_0^{\infty} j_{\ell}(kr) j_{\ell+m}(kr) g_{\ell,m}(kr') dk = \delta(r-r') \quad (1)$$

Then Ref. 1 gives formulas in terms of  ${}_1F_2$  functions for  $m = 0, 1, 2$ , and explicit results for  $g_{0,0}$ ,  $g_{1,0}$ ,  $g_{2,0}$ ,  $g_{0,1}$ , and  $g_{0,2}$ . The authors conjecture that the higher order inverse functions can be found by the methods presented in Ref. 1.

This problem has been treated in the classical literature. The inverse function  $g_{\ell,0}$  was found, in a form differing by an integration by parts, by Bateman,<sup>(2)</sup> and the higher order forms by Fox.<sup>(3)</sup> All these results are available in a standard text.<sup>(4)</sup> The general and quite simple formula is:

$$g_{\ell,m}(x) = \frac{8x^2}{\pi} \frac{d}{dx} x^2 n_{\ell}(x) j_{\ell+m}(x) \quad (2)$$

where  $n_{\ell}(x)$  denotes the spherical Neumann function. Equation (2) reduces to all the special cases treated in Ref. 1 except for  $g_{0,2}$ , for which it differs by a constant. Since it may be easily verified that  $j_{\ell}$  and  $j_{\ell+2}$  are themselves orthogonal over the range, the inverse function is not unique to an arbitrary additive constant. Furthermore, if the spherical Bessel functions are replaced by the corresponding expression in cylindrical Bessel functions, Eq. (2) is valid when  $\ell$  is not an integer.

To demonstrate the result of Eq. (2), the method of Ref. 4 will be followed. If two functions  $f(x)$  and  $g(x)$  satisfy the relation of Eq. (1), then  $g(x)$  is given in terms of the Mellin transform of  $f(x)$  by:\*

$$g(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} ds / F(1-s) \quad (3)$$

$$F(s) = \int_0^{\infty} x^{s-1} f(x) dx \quad (4)$$

The path of integration in Eq. (3) must lie in the strip where both  $F(s)$  and  $F(1-s)$  are analytic. In terms of cylindrical Bessel functions, using  $f(x)$  from Eq. (1), there results:

$$F(1-s) = \frac{\pi}{2} \int_0^{\infty} dx J_{\ell+1/2}(x) J_{\ell+m+1/2}(x) x^{-s-1} \quad (5)$$

This integral is a standard form,<sup>(5)</sup> and yields:

$$F(1-s) = \frac{\pi}{2^{s+2}} \frac{\Gamma(s+1) \Gamma(\ell + \frac{1}{2}m + \frac{1}{2} - \frac{1}{2}s)}{\Gamma(\frac{1}{2}s + \frac{1}{2}m + 1) \Gamma(\frac{1}{2}s + \ell + \frac{1}{2}m + \frac{3}{2}) \Gamma(\frac{1}{2}s - \frac{1}{2}m + 1)} \quad (6)$$

Thus,  $g_{\ell,m}$  is given by:

$$g_{\ell,m}(x) = \frac{8}{\pi} \frac{1}{2\pi i} \int ds \frac{2^{2s} \Gamma(s + \frac{1}{2}m + 1) \Gamma(s + \ell + \frac{1}{2}m + \frac{3}{2}) \Gamma(s - \frac{1}{2}m + 1)}{x^{2s} \Gamma(2s+1) \Gamma(\ell + \frac{1}{2}m + \frac{1}{2} - s)} \quad (7)$$

where  $s$  of Eq. (6) has been replaced by  $2s$ , and the abscissa of integration lies between  $-\frac{1}{2}$  and 0. Regardless of the parity of  $m$ , the poles

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\*Reference 4, p. 214.

of the first two gamma functions in the numerator are canceled by the poles of  $\Gamma(2s + 1)$  in the denominator. The path of integration may be closed by a large semicircle in the left half-plane, and the integral over the semicircle tends to zero. The third factor in the numerator has poles at  $s = -n - 1 + \frac{1}{2}m$ , where  $n$  takes on all positive integer values and zero. Those poles for which  $0 \leq n \leq \frac{1}{2}m - 1$  lie to the right of the integration path and do not contribute to the integral. Those poles for which  $\frac{1}{2}m \leq n \leq m - 1$  have vanishing residues. The residues of the remaining poles may be evaluated, and the reflection formula  $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$  may be used to simplify the expression. Thus:

$$g_{\ell, m}(x) = 4x^2(-1)^{\ell+m+1} \sum_m \frac{(-1)^n}{n!} \frac{\Gamma(2n+2-m) \left(\frac{1}{2}x\right)^{2n-m}}{\Gamma(n+1-m)\Gamma\left(n + \frac{1}{2} - \ell - m\right)\Gamma\left(n + \ell + \frac{3}{2}\right)} \quad (8)$$

$$= 4x^2(-1)^{\ell+1} \sum_0^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(2n+m+2) \left(\frac{1}{2}x\right)^{2n+m}}{\Gamma(n+m+1)\Gamma\left(n + \frac{1}{2} - \ell\right)\Gamma\left(n + \ell + m + \frac{3}{2}\right)} \quad (9)$$

Now we have the general formula\*

$$J_{\mu}(x)J_{\nu}(x) = \sum_0^{\infty} \frac{(-1)^n \Gamma(2n+\mu+\nu+1) \left(\frac{1}{2}x\right)^{2n+\mu+\nu}}{n! \Gamma(n+\mu+\nu+1)\Gamma(n+\mu+1)\Gamma(n+\nu+1)} \quad (10)$$

Setting  $\mu = -\ell - \frac{1}{2}$ ,  $\nu = \ell + m + \frac{1}{2}$ , this series becomes:

$$J_{-\ell-1/2}(x)J_{\ell+m+1/2}(x) = \sum_0^{\infty} \frac{(-1)^n \Gamma(2n+m+1) \left(\frac{1}{2}x\right)^{2n+m}}{n! \Gamma(n+m+1)\Gamma\left(n + \frac{1}{2} - \ell\right)\Gamma\left(n + \ell + m + \frac{3}{2}\right)} \quad (11)$$

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\*Reference 5, p. 147.

All the gamma functions in Eqs. (9) and (11) match except the one in the numerator. Multiply Eq. (11) by  $x$  and differentiate, and the two series become identical. Thus:

$$g_{\ell,m}(x) = 4x^2 (-1)^{\ell+1} \frac{d}{dx} x J_{-\ell-1/2}(x) J_{\ell+m+1/2}(x) \quad (12)$$

The identity  $N_{\ell+1/2}(x) = (-1)^{\ell+1} J_{-\ell-1/2}(x)$  and the return from cylindrical to spherical Bessel functions now yield Eq. (2). The analysis may be carried through in the same manner if  $\ell$  is not an integer. Only one of the three sets of numerator poles will cancel, and two series of the type of Eq. (9) result. Both may be identified as Bessel function products, and combined to yield the form of Eq. (2) in cylindrical Bessel functions.

For large  $x$ , Eq. (2) has the asymptotic form:

$$g_{\ell,m}(x) \rightarrow \frac{8x^2}{\pi} (-1)^{\ell+1} \cos\left(2x - \frac{m\pi}{2}\right) \quad (13)$$

which includes all the cases of Ref. 1. For small  $x$ , there results:

$$g_{\ell,m}(x) \rightarrow -\frac{(m+1)\Gamma\left(\ell + \frac{1}{2}\right) x^{m+2}}{\pi 2^{m-2} \Gamma\left(\ell + m + \frac{3}{2}\right)} \quad (14)$$

Since this vanishes for  $x = 0$ , the calculation of the potential from the phase shift, as discussed in Ref. 1, becomes much more practical. The structure of Eq. (2) makes it clear that the inverse function  $g_{\ell,m}(x)$  can always be written in terms of algebraic and trigonometric functions when  $\ell$  and  $m$  are both integers. If  $m$  is an even positive integer, an arbitrary constant may be added to  $g_{\ell,m}$ .

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